

Differentiability of Functions of Matrices

DIFFERENTIABILITY OF FUNCTIONS OF MATRICES

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Abstract. Let f be a function on the set of diagonal $n \times n$ matrices, and let \tilde{f} be the unique extension of f to the set of symmetric $n \times n$ matrices invariant with respect to conjugation by orthogonal matrices. We show that \tilde{f} has the same regularity properties as f . That is, if f is C^k , or $C^{k+\alpha}$, or C^∞ or C^ω then so is \tilde{f} .

It is well-known that every rotation-invariant function F on the space \mathcal{S} of real $d \times d$ symmetric matrices is determined by its restriction f to the diagonal matrices,

$$f(r_1, \dots, r_d) = F \begin{pmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & r_d \end{pmatrix}.$$

Since a 90° rotation in the ij -plane interchanges r_i and r_j , f must necessarily be symmetric,

$$f(r_{\sigma 1}, r_{\sigma 2}, \dots, r_{\sigma d}) = f(r_1, \dots, r_d), \quad \text{for all permutations } \sigma.$$

It is then natural to seek properties of f that are inherited by F .

For example, suppose f is a polynomial; then [5] $f = p(n_1, \dots, n_d)$ for some other polynomial p , where

$$n_k(r_1, \dots, r_d) = r_1^k + \dots + r_d^k, \quad k \geq 1,$$

are the Newton sums. It follows that

$$F(x) = p(\text{Trace}(x), \dots, \text{Trace}(x^d)), \quad x \in \mathcal{S},$$

since both sides are rotation-invariant and they agree on the diagonal matrices. Thus f polynomial implies F polynomial.

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Another interesting property is differentiability. Recently, Lewis and Sendov [7] showed that if f is C^1 or C^2 , then F is C^1 or C^2 respectively; moreover they derived formulas for $DF(x)$ and $D^2F(x)$ in terms of spectral quantities, i.e. the eigenvalues and eigenprojections of x . In this paper, we extend this result to C^n and derive a formula for $D^nF(x)$ in terms of the spectral quantities of x .

A theorem of C. Davis [1] asserts that f convex implies F convex, the canonical example being the negative of the logarithm of the determinant. There are several alternate proofs of this result, by Lewis [6], Rivin [8], and Grabovsky and Hijab [2]. As noted in [7], this convexity result, in the C^2 setting, is a consequence of the above differentiability result and the characterization of convexity in terms of nonnegativity of the second derivative.

These questions have natural generalizations in the context of compact Lie algebras. In this setting the issue is to identify the interesting properties that are inherited by an Ad-invariant function F on a compact Lie algebra \mathfrak{g} from its restriction f to a Cartan subalgebra \mathfrak{h} . The polynomial question in this setting is a theorem of Chevalley [4], and the convexity question was extended to this setting by Lewis [6] and subsequently by Grabovsky and Hijab [2].

For motivation, in §1 we derive the analog of this result in the radial setting and in §2 we derive the result in the context of symmetric matrices.

1. T R C

If $F : \mathbf{R}^d \rightarrow \mathbf{R}$ is continuous, then

$$(1) \quad f(r, \pi) = F(x) \quad x = r\pi,$$

is continuous on $\mathbf{R} \times \mathbf{S}^{d-1}$, even, and $f(0, \pi)$ does not depend on π . Conversely, if $f : \mathbf{R} \times \mathbf{S}^{d-1} \rightarrow \mathbf{R}$ is continuous, even, and $f(0, \pi)$ does not depend on π , then

$$(2) \quad F(x) = f\left(|x|, \frac{x}{|x|}\right), \quad x \neq 0,$$

extends to a continuous function on \mathbf{R}^d satisfying (1).

Let $\delta_\xi(\pi) = \xi - \langle \pi, \xi \rangle \pi$; then, for each ξ , the map $\delta_\xi : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a vector field tangent to \mathbf{S}^{d-1} . If f extends to a function on $\mathbf{R} \times \mathbf{R}^d$ that is polynomial in π , then so does $\delta_\xi(f)$. For f continuous in r and polynomial in π , define

$$\mathcal{L}_\xi^j(f)(r, \pi) = \langle \pi, \xi \rangle f(r, \pi) + \int_0^1 t^j \delta_\xi(f)(tr, \pi) dt.$$

Let f' denote the derivative with respect to r .

For $x \neq 0$, the maps $x \mapsto r = |x|$ and $x \mapsto \pi = x/r$ are analytic and their derivatives in the ξ direction are

$$r_\xi = \langle \pi, \xi \rangle, \quad \pi_\xi = \frac{\delta_\xi(\pi)}{r}.$$

By (1), $\delta_\xi(f)(0, \pi) = 0$ since $f(0, \pi)$ does not depend on π ; if f is C^1 in r and polynomial in π , (2) and the chain rule implies

(3)

$$\begin{aligned} D_\xi F(x) &= \langle \pi, \xi \rangle f'(r, \pi) + \frac{\delta_\xi(f)(r, \pi)}{r} \\ &= \langle \pi, \xi \rangle f'(r, \pi) + \int_0^1 \delta_\xi(f')(tr, \pi) dt = \mathcal{L}_\xi^0(f')(r, \pi), \quad x = r\pi \neq 0. \end{aligned}$$

If F is C^1 on \mathbf{R}^d , (3) is valid on \mathbf{R}^d ; if f is C^2 in r and polynomial in π , we may repeat this argument with $D_\xi F$ replacing F and $\mathcal{L}_\xi^0(f')$ replacing f ; we obtain $D_\xi^2 F(x) = \mathcal{L}_\xi^0(\mathcal{L}_\xi^0(f'))'(r, \pi)$ on $x = r\pi \neq 0$. If F is C^{n-1} on \mathbf{R}^d and f is C^n in r , we may continue in this manner to obtain

$$(4) \quad D_\xi^n F(x) = \mathcal{L}_\xi^0 \mathcal{L}_\xi^1 \dots \mathcal{L}_\xi^n (f^{(n)})(r, \pi), \quad x = r\pi \neq 0;$$

here we used $(\mathcal{L}_\xi^j f)' = \mathcal{L}_\xi^{j+1}(f')$.

Now suppose F is rotation-invariant; then $f = f(r)$ does not depend on π hence (4) implies

$$(5) \quad |D^n F(x)| \leq C \sup_{|r| \leq |x|} |f^{(n)}(r)|, \quad |x| \neq 0.$$

T 1. Let $n \geq 0$ and let F be a rotation-invariant function on \mathbf{R}^d . If the restriction of F to an axis is C^n , then F is C^n on \mathbf{R}^d .

The proof here mimics that of the matrix case in the next section; a simpler proof is possible.

Proof. Since F is continuous on \mathbf{R}^d , we may derive this by induction, so we may assume F is C^{n-1} on \mathbf{R}^d . Since f is even, $p(r) = f^{(n)}(0)r^n/n!$ either vanishes or is an even-order polynomial; hence $P(x) = p(|x|)$ is a polynomial on \mathbf{R}^d . Replacing F by $F - P$, we may further assume $f^{(n)}(0) = 0$. In this case, by (5), we conclude $D^n F(x) \rightarrow 0$ as $|x| \rightarrow 0$. Since F is C^{n-1} on \mathbf{R}^d and C^n away from the origin, this implies F is C^n on \mathbf{R}^d . \square

2. T M C

Let \mathcal{S} denote the vector space of real $d \times d$ symmetric matrices and let G be the group of $d \times d$ rotation matrices. A function $F : \mathcal{S} \rightarrow \mathbf{R}$ is *rotation-invariant* if $F(gxg^{-1}) = F(x)$ for every $x \in \mathcal{S}$ and $g \in G$. The result is

T 2. Let $n \geq 0$ and let F be a rotation-invariant function on \mathcal{S} . If the restriction of F to the diagonal matrices \mathcal{D} is C^n , then F is C^n on \mathcal{S} .

At the end of this section, we exhibit a formula (10) expressing $D^n F$ in terms of derivatives of the restriction.

C 1. Let F be a rotation-invariant function on \mathcal{S} and let f be its restriction to \mathcal{D} . If f is C^∞ on \mathcal{D} , then F is C^∞ on \mathcal{S} . If f is $C^{n,\alpha}$, $0 < \alpha < 1$, on \mathcal{D} , then F is $C^{n,\alpha}$ on \mathcal{S} . If f is analytic on \mathcal{D} , then F is analytic on \mathcal{S} .

The proof is at the end of the section.

A function F on a vector space sum $A \oplus B$ is $C^{n,N}$ if the partial derivatives $D_a^\alpha D_b^\beta F$ exist and are continuous on $A \oplus B$ for all multi-indices $|\alpha| \leq n$, $|\beta| \leq N$. The previous theorem is a special case of the slightly stronger

T 3. Let \mathcal{E} be a euclidean space and let $F : \mathcal{S} \oplus \mathcal{E} \rightarrow \mathbf{R}$ be rotation-invariant in the first variable. If the restriction of F to $\mathcal{D} \oplus \mathcal{E}$ is $C^{n,N}$, then F is $C^{n,N}$ on $\mathcal{S} \oplus \mathcal{E}$.

Let \mathcal{S}_0 denote the traceless matrices in \mathcal{S} , and let $\mathcal{D}_0 = \mathcal{D} \cap \mathcal{S}_0$. Since $\mathcal{S} = \mathcal{S}_0 \oplus \mathbf{R}$ and the trace is unchanged under conjugation by elements of G , this in turn follows from

T 4. Let \mathcal{E} be a euclidean space and let $F : \mathcal{S}_0 \oplus \mathcal{E} \rightarrow \mathbf{R}$ be rotation-invariant in the first variable. If the restriction of F to $\mathcal{D}_0 \oplus \mathcal{E}$ is $C^{n,N}$, then F is $C^{n,N}$ on $\mathcal{S}_0 \oplus \mathcal{E}$.

We turn to the proof of Theorem 4. To simplify notation, we now drop the subscript 0, i.e. henceforth the spaces of traceless symmetric and diagonal matrices will be denoted \mathcal{S} and \mathcal{D} respectively. We will argue by induction over (d, n, N) , where we impose the lexicographic ordering on triples (d, n, N) . Thus we assume the result is true for all dimensions lower than d and all orders of differentiability in n and N , and we assume the result is true for dimension d and all orders of differentiability on \mathcal{E} and lower than n on \mathcal{S} . Let \mathcal{S}^* be the open set of nonzero traceless symmetric matrices.

Let $x_0 \in \mathcal{S}^*$. Since conjugation by a rotation is an invertible analytic map on \mathcal{S} , we may assume that x_0 is diagonal with the diagonal

entries of x_0 arranged in decreasing order. Let $r_i(x_0)$, $i = 1, \dots, d$, be the diagonal entries of x_0 .

Let \mathfrak{g} be the vector space of real $d \times d$ skew-symmetric matrices; then $\mathcal{S} \oplus \mathfrak{g}$ is the vector space of all real $d \times d$ traceless matrices. Let $(\mathcal{S} \oplus \mathfrak{g})_0$ be the subspace of matrices x commuting with x_0 , $xx_0 = x_0x$. Then $x \in (\mathcal{S} \oplus \mathfrak{g})_0$ iff $x_{ij} = 0$ whenever $r_i(x_0) \neq r_j(x_0)$. Let $G_0 = G \cap (\mathcal{S} \oplus \mathfrak{g})_0$, $\mathcal{S}_0 = \mathcal{S} \cap (\mathcal{S} \oplus \mathfrak{g})_0$, and $\mathfrak{g}_0 = \mathfrak{g} \cap (\mathcal{S} \oplus \mathfrak{g})_0$. Then G_0 is the isotropy group of x_0 under the conjugation action, $(\mathcal{S} \oplus \mathfrak{g})_0 = \mathcal{S}_0 \oplus \mathfrak{g}_0$, and matrices in \mathcal{S}_0 , \mathfrak{g}_0 , and G_0 are block-diagonal with the same block structure. If $[x, y] = xy - yx$ is the usual bracket, then $\mathfrak{g}_0 = [\mathcal{S}_0, \mathcal{S}_0]$ and \mathcal{S}_0 is the orbit of \mathcal{D} under conjugation by matrices in G_0 .

L 1. F is $C^{n,N}$ on $\mathcal{S}_0 \oplus \mathcal{E}$.

Proof. We derive this by applying the inductive hypothesis block-by-block. Let $\mathcal{S}_0^{(k)}$ be the vector space of traceless symmetric matrices in \mathcal{S}_0 where all blocks, except possibly the k -th block, vanish, and let $\mathcal{D}^{(k)} = \mathcal{S}_0^{(k)} \cap \mathcal{D}$. Then $\mathcal{S}_0^{(1)} \oplus (\mathcal{D} \ominus \mathcal{D}^{(1)}) \oplus \mathcal{E}$ consists of block-diagonal matrices that are diagonal in all but the first block. Since the dimensions of the first block are strictly less than d and F is $C^{n,N}$ on \mathcal{D} , the inductive hypothesis implies F is $C^{n,N}$ on $\mathcal{S}_0^{(1)} \oplus (\mathcal{D} \ominus \mathcal{D}^{(1)}) \oplus \mathcal{E}$. More precisely, if $\mathcal{E}^{(1)} = (\mathcal{D} \ominus \mathcal{D}^{(1)}) \oplus \mathcal{E}$, then F is $C^{n,N}$ on $\mathcal{D} \oplus \mathcal{E} = \mathcal{D}^{(1)} \oplus \mathcal{E}^{(1)}$. Since the matrices in $\mathcal{D}^{(1)}$ are strictly smaller than $d \times d$, by the inductive hypothesis, F is $C^{n,N}$ on

$$\mathcal{S}_0^{(1)} \oplus \mathcal{E}^{(1)} = \mathcal{S}_0^{(1)} \oplus (\mathcal{D} \ominus \mathcal{D}^{(1)}) \oplus \mathcal{E}.$$

If $\mathcal{E}^{(2)} = \mathcal{S}_0^{(1)} \oplus (\mathcal{D} \ominus (\mathcal{D}^{(1)} \oplus \mathcal{D}^{(2)})) \oplus \mathcal{E}$, decomposing $\mathcal{S}_0^{(1)} \oplus (\mathcal{D} \ominus \mathcal{D}^{(1)}) \oplus \mathcal{E}$ into $\mathcal{D}^{(2)} \oplus \mathcal{E}^{(2)}$ and applying the inductive hypothesis again, F is $C^{n,N}$ on

$$\mathcal{S}_0^{(2)} \oplus \mathcal{E}^{(2)} = (\mathcal{S}_0^{(1)} \oplus \mathcal{S}_0^{(2)}) \oplus (\mathcal{D} \ominus (\mathcal{D}^{(1)} \oplus \mathcal{D}^{(2)})) \oplus \mathcal{E}.$$

Continuing in this manner, we conclude F is $C^{n,N}$ on

$$\mathcal{S}_0 \oplus \mathcal{E} = (\mathcal{S}_0^{(1)} \oplus \mathcal{S}_0^{(2)} \oplus \dots) \oplus \mathcal{E}$$

after finitely many steps. \square

Note this Lemma fails when $x_0 = 0$, since then $\mathcal{S}_0 = \mathcal{S}$.

L 2. There is a neighborhood U of x_0 in \mathcal{S} and an analytic map $X : U \rightarrow \mathfrak{g}_0^\perp$ such that $e^{X(x)}xe^{-X(x)}$ lies in \mathcal{S}_0 for $x \in U$.

Proof. For $x \in \mathcal{S} \oplus \mathfrak{g}$, let $\text{ad}(x) : \mathcal{S} \oplus \mathfrak{g} \rightarrow \mathcal{S} \oplus \mathfrak{g}$ be bracketing with x , $\text{ad}(x)(y) = [x, y]$. Then $\text{ad}(x)$ preserves the decomposition $\mathcal{S} \oplus \mathfrak{g}$ if

$x \in \mathfrak{g}$ and reverses it if $x \in \mathcal{S}$. If $\langle x, y \rangle = \text{Trace}(xy^t)$ is the usual inner product on $\mathcal{S} \oplus \mathfrak{g}$, then

$$\langle [x, y], z \rangle = \langle x, [z, y] \rangle$$

when $x, y \in \mathcal{S}$ and $z \in \mathfrak{g}$. This implies that the adjoint of $\text{ad}(x) : \mathcal{S} \rightarrow \mathfrak{g}$ is $-\text{ad}(x) : \mathfrak{g} \rightarrow \mathcal{S}$ when $x \in \mathcal{S}$.

Let \mathfrak{g}_0^\perp denote the orthogonal complement of \mathfrak{g}_0 in \mathfrak{g} and let \mathcal{S}_0^\perp denote the orthogonal complement of \mathcal{S}_0 in \mathcal{S} . Since the null-space of $\text{ad}(x_0) : \mathcal{S} \rightarrow \mathfrak{g}$ equals \mathcal{S}_0 , it follows that the range of $\text{ad}(x_0) : \mathfrak{g} \rightarrow \mathcal{S}$ is \mathcal{S}_0^\perp . Since $[\mathfrak{g}_0, x_0] = 0$, we conclude $[\mathfrak{g}_0^\perp, x_0] = \mathcal{S}_0^\perp$.

Define a map $\mathfrak{g}_0^\perp \oplus \mathcal{S}_0 \rightarrow \mathcal{S}$ by

$$(X, x) \mapsto e^{-X} x e^X.$$

At $(0, x_0)$, the derivative of this map is the linear map $(X, x) \mapsto (-[X, x_0]) \oplus x$, whose range equals $\mathcal{S}_0^\perp \oplus \mathcal{S}_0 = \mathcal{S}$. Thus the map is a diffeomorphism at $(0, x_0)$ onto a neighborhood U of x_0 in \mathcal{S} ; inverting this map, the result follows. \square

L **3.** F is $C^{n,N}$ on $\mathcal{S}^* \times \mathcal{E}$.

Proof. Combining the two previous lemmas shows F is $C^{n,N}$ on $U \times \mathcal{E}$ hence on $\mathcal{S}^* \times \mathcal{E}$. \square

At this point that we are left with establishing smoothness near $x_0 = 0$; this case is more significant than at first appears as the proof of Lemma 1 shows that the zero matrix “propagates” into larger and larger subspaces of \mathcal{S} . Nevertheless, we may be more specific about the asymptotic behavior of $F|_{\mathcal{D} \oplus \mathcal{E}}$ at the zero matrix:

L **4.** Without loss of generality, we may assume in addition that $D^k(F|_{\mathcal{D} \oplus \mathcal{E}}) = o(|x|^{n-k})$ as $x \rightarrow 0$ in \mathcal{D} for $0 \leq k \leq n$.

Proof. Let t be the n -th order Taylor polynomial of $F|_{\mathcal{D} \oplus \mathcal{E}}$ centered at $x_0 = 0$. Since F is rotation-invariant, $F|_{\mathcal{D} \oplus \mathcal{E}}$ is permutation-invariant, hence [5] there is a $C^{\infty,N}$ function p on $\mathcal{D} \oplus \mathcal{E}$, polynomial on \mathcal{D} , such that $t = p \circ n$, where $n = (n_1, \dots, n_d)$ are the Newton sums. Since the Newton sums extend to polynomial functions on $\mathcal{S} \oplus \mathcal{E}$, t extends to a polynomial function T on $\mathcal{S} \oplus \mathcal{E}$; replacing F by $F - T$, we are done. \square

To establish smoothness at the origin, we derive a representation formula for $D^n F$ in terms of derivatives of $F|_{\mathcal{D} \oplus \mathcal{E}}$, which is also of independent interest. This representation formula involves passing from the coordinate $x \in \mathcal{S}$ to “polar coordinates” (r, π) with $r \in \mathcal{D}$ in a manner analogous to that presented in the previous section.

A *projection* is a real $d \times d$ symmetric matrix π satisfying $\pi^2 = \pi$, and a *flag* is a d -tuple $\pi = (\pi_1, \dots, \pi_d)$ of *one-dimensional* projections that are mutually orthogonal, $\pi_i \pi_j = 0$ for $i \neq j$, and sum to the identity $\sum_i \pi_i = I$. Since $\text{Trace}(\pi_i) = 1$, π_i is not in \mathcal{S} .

Given $x \in \mathcal{S}$, let r_1, \dots, r_d denote its eigenvalues, listed with multiplicity, and let π_1, \dots, π_d denote the projections onto a corresponding orthonormal basis of eigenvectors. Then $r = (r_1, \dots, r_d)$ is in the space \mathbf{R}_0^d of vectors satisfying $r_1 + \dots + r_d = 0$ and $\pi = (\pi_1, \dots, \pi_d)$ is a flag. Conversely, if $r \in \mathbf{R}_0^d$ and π is a flag,

$$(6) \quad x = \sum_i r_i \pi_i$$

is in \mathcal{S} . It is easy to see that the set $\mathcal{F} \subset \mathcal{S}^d$ of flags is a compact metric space.

We say a flag $\pi = (\pi_1, \dots, \pi_d)$ is an *eigenflag* of x if (6) holds for some vector r ; this happens iff $x\pi_i = \pi_i x = r_i \pi_i$ for $i = 1, \dots, d$.

Let $(\mathbf{R}^d)'$ denote the open dense subset of vectors in \mathbf{R}^d with distinct entries and let \mathcal{S}' denote the subset of traceless symmetric matrices with distinct eigenvalues.

Let $r = r(x)$ equal to the vector of eigenvalues of $x \in \mathcal{S}$, arranged in decreasing order; using the compactness of \mathcal{F} , it follows easily that $r : \mathcal{S} \rightarrow \mathbf{R}_0^d$ is continuous. If $x \in \mathcal{S}'$, the corresponding eigenflag $\pi = \pi(x)$ is uniquely determined; this is not so if x has repeated eigenvalues. We claim the maps $x \mapsto r_i(x)$, $x \mapsto \pi_i(x)$ are analytic on \mathcal{S}' , and we compute the derivatives $r_{i\xi}$ and $\pi_{i\xi}$, $i = 1, \dots, d$, in the direction of $\xi \in \mathcal{S}$; this is a standard computation [3].

Let $n_k : \mathbf{R}^d \rightarrow \mathbf{R}$ be the k -th newton sum, $n_k(r) = (r_1^k + \dots + r_d^k)/k$, and let $n : \mathbf{R}^d \rightarrow \mathbf{R}^d$ be $n = (n_1, \dots, n_d)$. Also define $n : \mathcal{S} \rightarrow \mathbf{R}^d$ by $n = (n_1, \dots, n_d)$ with $n_k(x) = \text{Trace}(x^k)/k$. Then $n(x) = n(r(x))$. Since

$$\det(Dn(r)) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_d \\ & & \ddots & \\ r_1^{d-1} & r_2^{d-1} & \dots & r_d^{d-1} \end{pmatrix}$$

is the Vandermonde determinant, $n : (\mathbf{R}^d)' \rightarrow \mathbf{R}^d$ is a local diffeomorphism. Since $r = n^{-1}(n(r)) = n^{-1}(n(x))$, we conclude r is analytic on \mathcal{S}' .

L 5. For $\xi \in \mathcal{S}$, we have

$$(7) \quad r_{i\xi} = \langle \pi_i, \xi \rangle, \quad \pi_{i\xi} = \sum_{j \neq i} \frac{\pi_j \xi \pi_i + \pi_i \xi \pi_j}{r_i - r_j},$$

$i = 1, \dots, d$, on \mathcal{S}' .

Proof. By the chain rule,

$$Dr(x) = D(n^{-1} \circ n)(x) = D(n^{-1})(n(r)) \cdot Dn(x) = (Dn(r))^{-1} \cdot Dn(x).$$

Since $Dn_k(x) = x^{k-1}$, Cramer's rule yields

$$Dr(x) = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \dots \\ \pi_d \end{pmatrix},$$

or, what is the same,

$$(8) \quad r_{i\xi} = \langle \pi_i, \xi \rangle, \quad i = 1, \dots, d.$$

In particular, since r_i is analytic, this shows that the maps π_i , $i = 1, \dots, d$, are analytic.

To compute $\pi_{i\xi}$, differentiate $x\pi_i = \pi_i x = r_i \pi_i$ to get $\xi \pi_i + x \pi_{i\xi} = r_{i\xi} \pi_i + r_i \pi_{i\xi}$. Left multiply by π_j , $j \neq i$, to get

$$\pi_j \xi \pi_i + r_j \pi_j \pi_{i\xi} = \pi_j \xi \pi_i + \pi_j x \pi_{i\xi} = \pi_j r_{i\xi} \pi_i + \pi_j r_i \pi_{i\xi} = r_i \pi_j \pi_{i\xi}$$

which yields

$$\pi_j \pi_{i\xi} = \frac{\pi_j \xi \pi_i}{r_i - r_j} = \pi_j \pi_{i\xi} \pi_i.$$

Differentiating $\pi_i^2 = \pi_i$, we obtain $\pi_i \pi_{i\xi} + \pi_{i\xi} \pi_i = \pi_{i\xi}$, hence $\pi_i \pi_{i\xi} \pi_i = 0$; summing over j , we conclude

$$\pi_{i\xi} \pi_i = \sum_{j \neq i} \frac{\pi_j \xi \pi_i}{r_i - r_j}.$$

Adding this last equation to its transpose, we arrive at (7). □

We say a function $f : \mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E} \rightarrow \mathbf{R}$ is *symmetric* if

$$f(r_{\sigma 1}, \dots, r_{\sigma d}, \pi_{\sigma 1}, \dots, \pi_{\sigma d}, v) = f(r_1, \dots, r_d, \pi_1, \dots, \pi_d, v)$$

holds on $\mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E}$ for every permutation σ . A subset K is *symmetric* if $\mathbf{1}_K$ is symmetric.

We say f is *consistent* if $f(r, \pi, v) = f(r, \pi', v)$ whenever $x = \sum_i r_i \pi_i = \sum_i r_i \pi'_i$. This is the same as saying $f(r, \pi, v) = f(r, \pi', v)$ whenever $\pi_\lambda = \pi'_\lambda$, where

$$\pi_\lambda = \sum_{r_i = \lambda} \pi_i$$

for every eigenvalue λ of x .

If $F : \mathcal{S} \times \mathcal{E} \rightarrow \mathbf{R}$ is a continuous function, then $f : \mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E} \rightarrow \mathbf{R}$ defined by

$$(9) \quad f(r, \pi, v) = F\left(\sum_i r_i \pi_i, v\right), \quad r \in \mathbf{R}_0^d, \pi \in \mathcal{F}, v \in \mathcal{E},$$

is clearly continuous, symmetric and consistent. Note that F is rotation-invariant iff f does not depend on π .

L **6.** *If $f : \mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E} \rightarrow \mathbf{R}$ is continuous, symmetric, and consistent, there exists a unique continuous $F : \mathcal{S} \times \mathcal{E} \rightarrow \mathbf{R}$ satisfying (9).*

Proof. Since $r_i, \pi_i, i = 1, \dots, d$, are analytic on \mathcal{S}' and f is symmetric, it is clear that (9) defines F uniquely and continuously on $\mathcal{S}' \times \mathcal{E}$. If $x_n \in \mathcal{S}'$ and $x_n \rightarrow x \in \mathcal{S}$ and $v_n \rightarrow v$ in \mathcal{E} , we need to establish the convergence of $(F(x_n, v_n))$. To this end, let r_n denote the corresponding vectors of eigenvalues, arranged in decreasing order, and let π_n denote the corresponding eigenflags. Then r_n converges to the vector r of eigenvalues of x arranged in non-increasing order. If π is a limit point of (π_n) , then π is an eigenflag of x . By consistency, $f(r, \pi, v)$ depends only on r and the projections π_λ onto the λ -eigenspaces of x , hence only on x . Thus $f(r, \pi, v)$ does not depend on the subsequence, $(f(r_n, \pi_n, v_n)) = (F(x_n, v_n))$ converges to a limit, and (9) holds at all (r, π, v) . \square

Let $\mathcal{S}^d = \mathcal{S} \times \mathcal{S} \cdots \times \mathcal{S}$ be the d -fold product. Given $\xi \in \mathcal{S}$ and a skew-symmetric $d \times d$ matrix a , define a map $\delta = \delta(a, \xi) : \mathcal{S}^d \rightarrow \mathcal{S}^d$ by

$$\delta(a, \xi)(\pi)_i = \pi_i \xi \left(\sum_j a_{ij} \pi_j \right) + \left(\sum_j a_{ij} \pi_j \right) \xi \pi_i, \quad i = 1, \dots, d.$$

L **7.** *The map δ restricted to \mathcal{F} is a vector field tangent to \mathcal{F} .*

Proof. To see this, let $\pi(t) \in \mathcal{S}^d$ be a smooth curve of d -tuples of symmetric matrices starting at $\pi(0) \in \mathcal{F}$ satisfying $\dot{\pi}_i = \delta_i(\pi)$, $i = 1, \dots, d$, for t small. We show $\pi(t) \in \mathcal{F}$ by showing

- (1) $\sum_i \pi_i(t) = 1$,
- (2) $\pi_i(t) \pi_j(t) = 0, i \neq j$,
- (3) $\pi_i(t)^2 = \pi_i(t)$.

(1) follows since $\sum_i \dot{\pi}_i(t) = \sum_i \delta_i(\pi(t)) = 0$. Differentiation shows that $x_{ij}(t) = \pi_i(t) \pi_j(t)$, $i \neq j$, satisfies a linear system of differential equations with time-varying coefficients; since $x_{ij}(0) = 0, i \neq j$, (2) follows. (3) follows since by (2) $(d/dt) \pi_i(t)^2 = \dot{\pi}_i(t) \pi_i(t) + \pi_i(t) \dot{\pi}_i(t) = \dot{\pi}_i(t)$. Thus $\pi(t) \in \mathcal{F}$. \square

Define vector fields $\delta_{ij\xi} = -\delta_{ji\xi}$, $i \neq j$, on \mathcal{F} by

$$\delta(a, \xi) = \sum_{i \neq j} a_{ij} \delta_{ij\xi}.$$

Then for each i, j, ξ , $\delta_{ij\xi}$ is a vector field on \mathcal{F} . Note that (7) can be rewritten as

$$\pi_\xi = \frac{1}{2} \sum_{i \neq j} \frac{\delta_{ij\xi}(\pi)}{r_i - r_j} = \frac{1}{2} \delta(\rho, \xi),$$

where ρ is the skew-symmetric matrix with entries $1/(r_i - r_j)$.

Let $(\mathbf{R}_0^d)'$ denote the vectors in \mathbf{R}_0^d with distinct entries and let $(\mathbf{R}_0^d)^*$ be the nonzero vectors in \mathbf{R}_0^d .

If $f : \mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E} \rightarrow \mathbf{R}^d$ is polynomial in π , $f = (f_1, \dots, f_d)$, let

$$\mathcal{L}_\xi(f)(r, \pi, v) = \sum_i f_i \cdot \langle \pi_i, \xi \rangle + \frac{1}{4} \sum_{i \neq j} \int_0^1 \delta_{ij\xi}(f_i - f_j)(r(t), \pi, v) dt.$$

If f is $C^{n,N}$ on $\mathbf{R}_0^d \oplus \mathcal{E}$ and polynomial in π , so is $\mathcal{L}_\xi(f)$.

Let $Df = (f_{r_1}, \dots, f_{r_d})$ be the gradient of f in r .

L **8.** If f given by (9) is $C^{n,N}$ on $\mathbf{R}_0^d \oplus \mathcal{E}$ and polynomial in π , then

$$(10) \quad D_\xi^n F \left(\sum_i r_i \pi_i, v \right) = (\mathcal{L}_\xi D)^n f(r, \pi, v)$$

on $(\mathbf{R}_0^d)' \times \mathcal{F} \times \mathcal{E}$.

Proof. Let $\pi(t)$ be the integral curve of $\delta_{ij\xi}$ starting from $\pi \in \mathcal{F}$; since the sum of i -th and j -th components of $\delta_{ij\xi}$ vanishes, $\pi_i(t) + \pi_j(t)$ does not depend on t ; then f consistent and polynomial in π and $r_i = r_j$ implies $f(r, \pi(t), v)$ does not depend on t , hence $\delta_{ij\xi}(f)(r, \pi) = 0$.

Given $r \in \mathbf{R}^d$, let $r(t)$ differ from r only in the i -th and j -th components, by setting $r_i(t) = tr_i + (1-t)(r_i + r_j)/2$, $r_j(t) = tr_j + (1-t)(r_i + r_j)/2$.

Since $\delta_{ij\xi}(f)(r(0), \pi, v)$ vanishes, the fundamental theorem of calculus applied to $\delta_{ij\xi}(f)(r(t), \pi, v)$ implies

$$(11) \quad \frac{\delta_{ij\xi}(f)(r, \pi, v)}{r_i - r_j} = \frac{1}{2} \int_0^1 \delta_{ij\xi}(f_{r_i} - f_{r_j})(r(t), \pi, v) dt.$$

By the chain rule, (11), and (7),

$$(12) \quad \begin{aligned} D_\xi F \left(\sum_i r_i \pi_i, v \right) &= \sum_i f_{r_i} \cdot \langle \pi_i, \xi \rangle + \frac{1}{2} \sum_{i \neq j} \frac{\delta_{ij\xi}(f)}{r_i - r_j} \\ &= \mathcal{L}_\xi(D(f))(r, \pi, v) \end{aligned}$$

on $(\mathbf{R}_0^d)' \times \mathcal{F} \times \mathcal{E}$. If F is $C^{1,N}$, (12) is valid on $\mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E}$, thus

$$D_\xi^2 F \left(\sum_i r_i \pi_i, v \right) = (\mathcal{L}_\xi D)^2(f)(r, \pi, v)$$

on $(\mathbf{R}_0^d)' \times \mathcal{F} \times \mathcal{E}$. Since F is $C^{n-1,N}$ we may repeat this argument $n-1$ times; the result follows. \square

If F is rotation-invariant, then F is $C^{n,N}$ on $\mathcal{S}^* \oplus \mathcal{E}$, and hence (10) is valid on $(\mathbf{R}_0^d)' \times \mathcal{F} \times \mathcal{E}$. Moreover, $f = f(r, v)$ does not depend on π and hence (10) implies

$$(13) \quad |D^n F(x, v)| \leq C \sup_{|r| \leq |x|} |D^n f(r, v)|.$$

Recalling Lemma 4, this implies $D^n F(x, v) \rightarrow 0$ as $|x| = |r| \rightarrow 0$; since we know F is $C^{n-1,N}$, this implies $F(\cdot, v)$ is C^n on \mathcal{S} for each $v \in \mathcal{E}$. This in turn implies the validity of (10) on $\mathbf{R}_0^d \times \mathcal{F} \times \mathcal{E}$, which in turn implies F is $C^{n,N}$ on $\mathcal{S} \oplus \mathcal{E}$. This completes the proof of Theorem 4.

We now prove the Corollary. The first statement is an immediate consequence of Theorem 2. Away from the origin, if f is $C^{n,\alpha}$ or analytic, the proof (Lemmas 1, 2, 3) of Theorem 4, unchanged, establishes F is $C^{n,\alpha}$ or analytic respectively. If f is $C^{n,\alpha}$, then, from Theorem 2, F is C^n . If t_n is the n -th order Taylor polynomial of f at the origin, then t_n is permutation-invariant, hence t_n is the restriction to \mathcal{D} of a rotation-invariant polynomial T_n on \mathcal{S} . It follows that T_n is the n -th order Taylor polynomial of F at the origin. Replacing F by $F - T_n$, since $D^n t_n(r) = D^n f(0)$ and $D^n T_n(x) = D^n F(0)$, by (13) we have

$$|D^n F(x) - D^n F(0)| \leq C \sup_{|r| \leq |x|} |D^n f(r) - D^n f(0)|.$$

Thus F is $C^{n,\alpha}$ at the origin. If f is analytic at the origin, $|f(r) - t_n(r)| \leq C|r/2\epsilon|^n$ on $|r| < \epsilon$ for all n ; since F , T_n and $|x|^n$ are rotation-invariant, it follows that $|F(x) - T_n(x)| \leq C|x/2\epsilon|^n$ on $|x| < \epsilon$ for all n ; thus F is analytic at the origin.

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